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# Totally geodesic subgroups of diffeomorphisms

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**Abstract**

We determine the Riemannian manifolds for which the group of exact volume preserving diffeomorphisms is a totally geodesic subgroup of the group of volume preserving diffeomorphisms, considering right invariant $L^2$-metrics. The same is done for the subgroup of Hamiltonian diffeomorphisms as a subgroup of the group of symplectic diffeomorphisms in the Kähler case. These are special cases of totally geodesic subgroups of diffeomorphisms with Lie algebras big enough to detect the vanishing of a symmetric 2-tensor field. 

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## 1. Introduction

Euler equation for an ideal fluid flow $(\mathrm{d}/\mathrm{d}t)u = -\nabla_u u - \operatorname{grad} p$ is the geodesic equation on the group of volume preserving diffeomorphisms with right invariant $L^2$-metric, see [1].

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For a compact oriented surface this is the group of symplectic diffeomorphisms. The fact that the existence of a single valued stream function for the velocity field $u$ at the initial moment implies the existence of a single valued stream function at any other moment of time, is equivalent to the fact that the group of Hamiltonian diffeomorphisms is totally geodesic in the group of symplectic diffeomorphisms. We will prove that a closed, oriented surface having this property either has the first Betti number zero (the Lie algebras of symplectic and Hamiltonian vector fields coincide) or it is a flat 2-torus.

A much more general classification result is actually true: twisted products of a torus by a Riemannian (resp. Kähler) manifold with vanishing first Betti number are the only Riemannian (resp. Kähler) manifolds, where the exact volume preserving diffeomorphisms lie totally geodesic in the Lie group of volume preserving diffeomorphisms (resp. the Hamiltonian diffeomorphisms in the symplectic diffeomorphisms). More precisely:

**Theorem 1.** *Let $(M, g)$ be a closed, connected and oriented Riemannian manifold with volume form $\mu$. Then the following are equivalent:*

1. *The group of exact volume preserving diffeomorphisms is a totally geodesic subgroup in the group of all volume preserving diffeomorphisms.*
2. *Every harmonic 1-form is parallel.*
3. $\mathrm{ric}(\beta_1, \beta_2) = 0$ *for all harmonic 1-forms* $\beta_1, \beta_2$.
4. $(M, g)$ *is a twisted product of a flat torus $T^k = \mathbb{R}^k/\Lambda$ and a closed, connected, oriented Riemannian manifold $F$ with vanishing first Betti number, i.e. $M = \mathbb{R}^k \times_\Lambda F$.*
5. *For all 2-forms $\alpha$ and all harmonic 1-forms $\beta$ one has*

$$\int_M g(\mathrm{d}\delta\alpha, \delta\alpha \wedge \beta)\mu = 0.$$

This result corrects the Remark 3.1 in [2] that the subgroup of exact volume preserving diffeomorphisms is totally geodesic in the group of volume preserving diffeomorphisms on compact manifolds. In their book this remark was used for curvature calculations related to the diffeomorphism group of a flat torus, where such a statement indeed holds.

**Theorem 2.** *Let $(M, g, J, \omega)$ be a closed, connected Kähler manifold. Then the following are equivalent:*

1. *The group of Hamiltonian diffeomorphisms is a totally geodesic subgroup in the group of all symplectic diffeomorphisms.*
2. *Every harmonic 1-form is parallel.*
3. $\mathrm{ric}(\beta_1, \beta_2) = 0$ *for all harmonic 1-forms* $\beta_1, \beta_2$.
4. $(M, g, J, \omega)$ *is a twisted product of a flat torus and a closed connected Kähler manifold with vanishing first Betti number.*
5. *For all functions $f$ and all harmonic 1-forms $\beta$ one has*

$$\int_M (\Delta f)\mathrm{d}f \wedge \beta \wedge \omega^{n-1} = 0.$$

We also prove that there is no group of diffeomorphisms $G \subset \mathrm{Diff}(M)$ containing the group of volume preserving diffeomorphisms (for a manifold $M$ with volume form) or the group of symplectic diffeomorphisms (for an almost Kähler manifold) as totally geodesic subgroups. In particular the group of symplectic diffeomorphisms is not totally geodesic in the group of volume preserving diffeomorphisms for an almost Kähler manifold.

## 2. Geodesic equations

The problem can be formulated in the setting of regular Fréchet–Lie groups: given a regular Fréchet–Lie group in the sense of Kriegl–Michor, see [6], and a (bounded, positive definite) scalar product $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$, we can define a right invariant metric on $G$ by

$$G_x(\xi, \eta) := g((T_x\rho^x)^{-1}\xi, (T_x\rho^x)^{-1}\eta) \quad \text{for } \xi, \eta \in T_xG,$$

where $\rho^x$ denotes the right translation by $x$ on $G$. The energy functional of a smooth curve $c : \mathbb{R} \to G$ is defined by

$$E(c) = \int_a^b G_{c(t)}(c'(t), c'(t))\mathrm{d}t = \int_a^b g(\delta^r c(t), \delta^r c(t))\mathrm{d}t,$$

where $\delta^r$ denotes the right logarithmic derivative on the Lie group $G$.

Assuming $c : [a, b] \to G$ to be a geodesic with respect to the right invariant (weak) Riemannian metric, variational calculus yields

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -\mathrm{ad}(u)^{\mathrm{T}}u, \qquad u(t) = \delta^r c(t),$$

where $\mathrm{ad}(X)^{\mathrm{T}} : \mathfrak{g} \to \mathfrak{g}$ denotes the adjoint of $\mathrm{ad}(X)$ with respect to the Hilbert scalar product, see [6], which we assume to exist as bounded linear map $\mathrm{ad}(\cdot)^{\mathrm{T}} : \mathfrak{g} \to L(\mathfrak{g})$. A Lie subgroup $H \subseteq G$ is totally geodesic if any geodesic $c$ with $c(a) = e$ and $c'(a) \in \mathfrak{h}$ stays in $H$. This is the case if $\mathrm{ad}(X)^{\mathrm{T}}X \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. If there is a geodesic in $G$ in any direction of $\mathfrak{h}$, then the condition is necessary and sufficient.

The setting for the whole article is the following: given a regular Fréchet–Lie group $G$ with Lie algebra $\mathfrak{g}$ and a bounded, positive definite scalar product on $\mathfrak{g}$, we assume that $\mathrm{ad}(\cdot)^{\mathrm{T}} : \mathfrak{g} \to L(\mathfrak{g})$ exists and is bounded. Furthermore we are given a splitting subalgebra $\mathfrak{h}$, i.e. $\mathfrak{h}$ has an orthogonal complement $\mathfrak{h}^\perp$ in $\mathfrak{g}$ with respect to the scalar product. We only assume $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as orthogonal direct sum, in the algebraic sense. It follows that $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are closed, and that the orthogonal projections onto $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are bounded with respect to the Fréchet space topology. $\mathfrak{h}$ is called *totally geodesic* in $\mathfrak{g}$ if $\mathrm{ad}(X)^{\mathrm{T}}X \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. The following reformulation of the condition provides the main condition for our work.

**Lemma 1.** *In the situation above,* $\mathfrak{h}$ *is totally geodesic in* $\mathfrak{g}$ *iff* $\langle [X, Y], X\rangle = 0$ *for all* $X \in \mathfrak{h}$ *and* $Y \in \mathfrak{h}^\perp$.

In this article we consider the following important examples of the outlined situation: let $M$ be a closed, connected and oriented manifold. The regular Fréchet–Lie group $\mathrm{Diff}(M)$

is modeled on the vector fields $\mathfrak{X}(M)$, a Fréchet space. The Lie algebra is $\mathfrak{X}(M)$ with the negative of the usual Lie bracket. The symbol $[\cdot,\cdot]$ will denote the usual Lie bracket and $\mathrm{ad}(X)Y = -[X, Y]$. The following subgroups are regular Fréchet–Lie subgroups of Diff($M$), see [4]:

1. The group $\mathrm{Diff}(M, \mu)$ of volume preserving diffeomorphisms of $(M, \mu)$, where $\mu$ is a volume form on $M$; its Lie algebra is $\mathfrak{X}(M, \omega)$, the Lie algebra of divergence free vector fields.
2. The group $\mathrm{Diff}_{\mathrm{ex}}(M, \mu)$ of exact volume preserving diffeomorphisms of $(M, \mu)$ with Lie algebra $\mathfrak{X}_{\mathrm{ex}}(M, \mu) = \{X : i_X\mu \text{ exact differential form}\}$.
3. The group $\mathrm{Diff}(M, \omega)$ of symplectic diffeomorphisms of the symplectic manifold $(M, \omega)$ with Lie algebra $\mathfrak{X}(M, \omega)$, the Lie algebra of symplectic vector fields.
4. The group $\mathrm{Diff}_{\mathrm{ex}}(M, \omega)$ of Hamiltonian diffeomorphisms of $(M, \omega)$ with Lie algebra $\mathfrak{X}_{\mathrm{ex}}(M, \omega)$, the Lie algebra of Hamiltonian vector fields.

Let $(M, g)$ denote a closed connected and orientable Riemannian manifold with Riemannian metric $g$, $\nabla$ the Levi–Civita covariant derivative and $\mu$ the canonical volume form on $M$ induced by the metric $g$ and a choice of orientation. By $\sharp_g : T^*M \to TM$ we denote the geometric lift $g(\sharp a_g, \alpha, \cdot) = \alpha$ and by $\flat_g$ its inverse. We will omit the index $g$ when no confusion is possible. The Hodge-$*$-operator is given with respect to the volume form $\mu$ such that $g(\beta, \eta)\mu = \beta \wedge *\eta$ for $\beta$, $\eta$ $k$-forms, where $g$ denotes the respective scalar product on the forms. The exterior derivative is denoted by $d$, the codifferential by $\delta = (-1)^{n(k+1)+1} * d*$ on $k$-forms. With this convention $d$ and $\delta$ are adjoint with respect to scalar product on forms. Furthermore $\Delta = d\delta + \delta d$.

In the case of $G = \mathrm{Diff}(M)$ the adjoint of $\mathrm{ad}(X)$, with respect to the induced right invariant $L^2$-structure, is given by the expression (see [7])

$$\mathrm{ad}(X)^{\mathrm{T}} X = \nabla_X X + (\mathrm{div}\, X)X + \tfrac{1}{2}\mathrm{grad}(g(X, X)). \tag{1}$$

We apply here the notions of gradient of a function $\mathrm{grad}\, f = \sharp(\mathrm{d}f)$ and divergence of a vector field $\mathrm{div}\, X = -\delta(\flat X)$, i.e. $L_X\mu = \mathrm{div}(X)\mu$. If $H \subseteq G$ is a subgroup with splitting subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, then the adjoint in $\mathfrak{h}$ of $\mathrm{ad}(X)$, for $X \in \mathfrak{h}$ is

$$\mathrm{ad}(X)|_{\mathfrak{h}}^{\mathrm{T}} X = \pi(\mathrm{ad}(X)|_{\mathfrak{g}}^{\mathrm{T}} X),$$

where $\pi : \mathfrak{g} \to \mathfrak{h}$ denotes the orthogonal projection.

In particular, by projecting (1) on $\mathfrak{X}(M, \mu)$, we obtain for the volume preserving diffeomorphisms $\mathrm{Diff}(M, \mu)$ the adjoint

$$\mathrm{ad}(X)|_{\mathfrak{X}(M,\mu)}^{\mathrm{T}} X = \nabla_X X + (\mathrm{grad}\, p)(X), \qquad \Delta p = \mathrm{div}(\nabla_X X).$$

Such a function $p$ exists, is unique up to a constant and smooth by application of the smooth inverse of the Laplacian on its range. Hence $(\mathrm{grad}\, p)(X)$ is a well defined smooth vector field. Remark, that the orthogonal complement to divergence free vector fields are gradients of some functions, with respect to the $L^2$-metric, which is easily seen due to the orthogonal Hodge decomposition $\mathfrak{X}(M) = \sharp_g\, \mathrm{d}\Omega^0(M) \oplus \sharp_g \ker \delta$, where $\sharp_g$ denotes the geometric lift.

Consequently the geodesic equation on $\mathrm{Diff}(M,\mu)$ with right invariant $L^2$-metric is

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -\nabla_u u - \operatorname{grad} p,$$

the Euler equation for an ideal fluid flow, see [1].

Let $M$ be a closed connected almost Kähler manifold $(M, g, \omega, J)$, i.e. the symplectic form $\omega$, the almost complex structure $J$ and the Riemannian metric $g$ satisfy the relation $g(X, Y) = \omega(X, JY)$. Note that a Kähler manifold has a natural orientation given by $J$. Moreover we have $\flat_\omega(X) = -(\flat_g X) \circ J = \flat_g(JX)$ and $\sharp_\omega\varphi = -J\sharp_g\varphi = \sharp_g(\varphi \circ J)$, especially $\sharp_\omega : T_x^*M \to T_xM$ is an isometry, for $J$ and $\sharp_g$.

For the symplectic diffeomorphisms $\mathrm{Diff}(M,\omega)$ we obtain the following adjoint

$$\mathrm{ad}(X)|^{\mathrm{T}}_{\mathfrak{X}(M,\omega)}X = \nabla_X X + \tfrac{1}{2}\operatorname{grad}(g(X, X)) + \sharp_\omega((\delta\alpha)(X)),$$
$$\mathrm{d}((\delta\alpha)(X)) = -\mathrm{d}i_{\nabla_X X} + \tfrac{1}{2}\operatorname{grad}(g(X, X))\omega,$$

by projecting (1) on $\mathfrak{X}(M,\omega)$. Indeed, via the symplectic lift the orthogonal Hodge decomposition of

$$\Omega^1(M) = \mathrm{d}\Omega^0(M) \oplus \mathcal{H}^1(M) \oplus \delta\Omega^2(M),$$

can be carried to the vector fields. Symplectic vector fields are those with $L_X\omega = \mathrm{d}(\flat_\omega X) = 0$, i.e. $\flat_\omega X \in \ker d = \mathrm{d}\Omega^0(M) \oplus \mathcal{H}^1(M)$. So there is some (symplectic) harmonic part and some Hamiltonian part, see [4]. In the above formula $(\delta\alpha)(X)$ is uniquely determined and smoothly dependent on $X \in \mathfrak{X}(M,\omega)$, by the Hodge decomposition. The divergence part is zero, since symplectic diffeomorphisms are volume preserving. Consequently the geodesic equation on $\mathrm{Diff}(M,\omega)$ with right invariant $L^2$-metric is

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -\nabla_u u - \frac{1}{2}\operatorname{grad}(g(u, u)) - \sharp_\omega(\delta\alpha).$$

## 3. Proof of the main results

In this section we develop the necessary notions and prove the main results of the article. These are Theorems 1 and 2 from Section 1, where we provide several equivalent conditions, geometric and analytic ones, in the Riemannian and Kähler case, such that the exact volume preserving diffeomorphisms lie totally geodesic in the group of volume preserving diffeomorphisms, respectively the Hamiltonian diffeomorphisms in the symplectic diffeomorphisms.

For a 1-form $\varphi$ we set

$$(\nabla\varphi)^{\mathrm{sym}}(X, Y) := (\nabla_X\varphi)(Y) + (\nabla_Y\varphi)(X),$$

the symmetric part of $\nabla\varphi$. Note that $\sharp\varphi$ is a Killing vector field, i.e. generates a flow of isometries, if and only if $(\nabla\varphi)^{\mathrm{sym}} = 0$. Note also, that

$$\mathrm{d}\varphi(X, Y) = (\nabla_X\varphi)(Y) - (\nabla_Y\varphi)(X),$$

the skew symmetric part of $\nabla\varphi$, and $\mathrm{tr}(\nabla\varphi)^{\mathrm{sym}} = 2\operatorname{div}\sharp\varphi = -2\delta\varphi$.

**Lemma 2.** *Let $(M, g)$ be a closed oriented $n$-dimensional Riemannian manifold and $\mathfrak{g} \subseteq \mathfrak{X}(M)$ a closed subalgebra, such that $\mathrm{ad}(X)^{\mathrm{T}} : \mathfrak{g} \to \mathfrak{g}$ exists for all $X \in \mathfrak{g}$. Then one has*

$$2 \int_M g(\mathrm{ad}(X)^{\mathrm{T}}(X), Y)\mu = \int_M ((\nabla \flat Y)^{\mathrm{sym}} + (\mathrm{div}\, Y) g)(X, X)\mu.$$

*Moreover*

$$\mathrm{tr}((\nabla \flat Y)^{\mathrm{sym}} + (\mathrm{div}\, Y) g) = (n+2)\mathrm{div}\, Y.$$

*Especially the symmetric* 2*-tensor field* $(\nabla \flat Y)^{\mathrm{sym}} + (\mathrm{div}\, Y) g = 0$ *iff* $(\nabla \flat Y)^{\mathrm{sym}} = 0$, *i.e.* $Y$ *is Killing*.

**Proof.** We have:

$$\begin{aligned}\int g(\mathrm{ad}(X)^{\mathrm{T}}(X), Y)\mu &= \int g(X, [X, Y])\mu = \int (g(X, \nabla_X Y) - g(X, \nabla_Y X))\mu \\ &= \int \left( i_X \nabla_X \flat Y - \frac{1}{2} L_Y g(X, X) \right) \mu \\ &= \int \frac{1}{2} (\nabla \flat Y)^{\mathrm{sym}}(X, X)\mu + \frac{1}{2} g(X, X) L_Y \mu \\ &= \frac{1}{2} \int \left( (\nabla \flat Y)^{\mathrm{sym}} + (\mathrm{div}\, Y) g \right)(X, X)\mu.\end{aligned}$$

The second statement follows from $\mathrm{tr}(\nabla(\flat Y)^{\mathrm{sym}}) = 2\mathrm{div}\, Y$ and $\mathrm{tr}(g) = n$. □

**Definition 1.** We say $\mathfrak{g} \subseteq \mathfrak{X}(M)$ is *big enough to detect the vanishing of a symmetric* 2*-tensor field*, if a symmetric 2-tensor field $T \in \Gamma(S^2 T^* M)$ vanishes if

$$\int_M T(X, Y)\mu = 0 \quad \text{for all } X, Y \in \mathfrak{g}.$$

**Remark 1.** Let $\mathfrak{g} \subseteq \mathfrak{X}(M)$ be the Lie algebra of a Lie group of diffeomorphisms $G$, such that $\mathrm{ad}(\cdot)^{\mathrm{T}} : \mathfrak{g} \to L(\mathfrak{g})$ is bounded. Suppose $H \subseteq G$ is a Lie subgroup with splitting Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and assume that $\mathfrak{h}$ is big enough to detect the vanishing of a symmetric 2-tensor. Under these conditions Lemma 2 gives a necessary and sufficient condition for $\mathfrak{h}$ to be totally geodesic in $\mathfrak{g}$, namely $(\nabla \flat Y)^{\mathrm{sym}} = 0$ for all $Y \in \mathfrak{h}^{\perp}$. In particular $Y$ is a Killing vector field for all $Y \in \mathfrak{h}^{\perp}$. Since the Lie algebra of Killing vector fields is finite dimensional, it follows that a Lie algebra $\mathfrak{h}$ big enough to detect the vanishing of a symmetric 2-tensor field, which is totally geodesic in $\mathfrak{g}$, must have finite codimension.

Next we give examples of Lie algebras of vector fields big enough to detect the vanishing of a symmetric 2-tensor field, needed for the proof of Theorems 1 and 2: the Lie algebra of Hamiltonian vector fields and the Lie algebra of exact divergence free vector fields.

**Lemma 3.** *Let $(M, \omega)$ be a symplectic manifold. Then the Lie algebra of compactly supported Hamiltonian vector fields is big enough to detect the vanishing of a symmetric* 2*-tensor field*.

**Proof.** Suppose $T$ is a symmetric 2-tensor field, which does not vanish at a point in $M$. Using Darboux's theorem, and rescaling $\omega$ and $T$ by constants, we may choose a chart $M \supseteq U \to (-1,1)^{2n} \subseteq \mathbb{R}^{2n}$, such that

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \cdots + \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n},$$

and $T_{22}(x) > 0$ for all $x \in (-1,1)^{2n}$, where $T = \sum T_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$. Now choose a bump function $b : \mathbb{R} \to [0,1]$, such that $b(t) = 0$ for $|t| \geq \frac{1}{2}$ and $b(0) = 1$. For $0 < \varepsilon \leq 1$ we define

$$\lambda_\varepsilon(x^1, \ldots, x^{2n}) := b\left(\frac{x^1}{\varepsilon}\right) b(x^2) \cdots b(x^{2n}),$$

and $Z_\varepsilon := \sharp_\omega \,\mathrm{d}\lambda_\varepsilon$. Since the support of $\lambda_\varepsilon$ is contained in $(-\varepsilon, \varepsilon) \times (-1,1)^{2n-1}$, $Z_\varepsilon$ extends by zero to a compactly supported Hamiltonian vector field on $M$. An easy calculation shows

$$\lim_{\varepsilon \to 0} \varepsilon \int_M T(Z_\varepsilon, Z_\varepsilon)\omega^n = \int_{(-1,1)^{2n}} (b'(x^1)b(x^2)\cdots b(x^{2n}))^2 T_{22}(0, x^2, \ldots, x^{2n})\omega^n > 0,$$

and hence $\int_M T(Z_\varepsilon, Z_\varepsilon)\omega^n \neq 0$ for $\varepsilon$ small enough. □

**Lemma 4.** *Let* $(M, \mu)$ *be a manifold with volume form,* $\dim(M) > 1$. *Then the Lie algebra of compactly supported exact divergence free vector fields is big enough to detect the vanishing of a symmetric* 2*-tensor field.*

**Proof.** As in the proof of Lemma 3, we choose a chart $M \supseteq U \to (-1,1)^n \subseteq \mathbb{R}^n$, such that

$$\mu = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

and such that $T_{22} > 0$ on $x \in (-1,1)^n$. Take a bump function $b$ as above and set

$$\lambda_\varepsilon := b\left(\frac{x^1}{\varepsilon}\right) b(x^2) \cdots b(x^n).$$

Now define $i_{Z_\varepsilon}\mu := \mathrm{d}(\lambda_\varepsilon \,\mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^n)$. Then $Z_\varepsilon$ is a compactly supported exact divergence free vector field on $M$ and

$$Z_\varepsilon = b\left(\frac{x^1}{\varepsilon}\right) b'(x^2)b(x^3)\cdots b(x^n)\frac{\partial}{\partial x^1} - \frac{1}{\varepsilon} b'\left(\frac{x^1}{\varepsilon}\right) b(x^2)\cdots b(x^n)\frac{\partial}{\partial x^2}.$$

Again we get

$$\lim_{\varepsilon \to 0} \varepsilon \int_M T(Z_\varepsilon, Z_\varepsilon)\mu = \int_{(-1,1)^n} (b'(x^1)b(x^2)\cdots b(x^n))^2 T_{22}(\varepsilon x^1, x^2, \ldots, x^n)\mu > 0,$$

and hence $\int_M T(Z_\varepsilon, Z_\varepsilon)\mu \neq 0$ for $\varepsilon$ small enough. □

**Definition 2** (twisted products). Let $T^k = \mathbb{R}^k/\Lambda$ be a flat torus, equipped with the metric induced from the Euclidean metric on $\mathbb{R}^k$. Suppose $F$ is an oriented Riemannian manifold and that $\Lambda$ acts on $F$ by orientation preserving isometries. The total space of the associated fiber bundle $\mathbb{R}^k \times_\Lambda F \to T^k$ is an oriented Riemannian manifold in a natural way. Locally over $T^k$ the metric is the product metric. We call every manifold obtained in this way a *twisted product* of a flat torus and the oriented Riemannian manifold $F$.

If $k$ is even, $F$ Kähler and $\Lambda$ acts by isometries preserving the Kähler structure then $\mathbb{R}^k \times_\Lambda F \to T^k$ is a Kähler manifold in a natural way and we call it a twisted product of a flat torus with the Kähler manifold $F$.

**Proof of Theorem 1.** Recall that the orthogonal complement of $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$ in $\mathfrak{X}(M,\mu)$ is $\{\sharp\beta : \beta \text{ harmonic } 1-\text{form}\}$. The equivalence (1) $\Leftrightarrow$ (2) now follows immediately from Remark 1, Lemma 4 and the fact that for closed 1-forms $(\nabla\beta)^{\mathrm{sym}} = 0$ is equivalent to $\nabla\beta = 0$.

(2) $\Rightarrow$ (3) is obvious from the definition of the curvature $R_{X,Y}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ and $\mathrm{ric} = -\mathrm{tr}_{13}R$.

The integrated Bochner equation on 1-forms, see for example [5], takes the form

$$\langle\Delta\alpha,\alpha\rangle = \|\nabla\alpha\|^2 + \int_M \mathrm{ric}(\alpha,\alpha)\mu,$$

and (3) $\Rightarrow$ (2) follows.

(4) $\Rightarrow$ (2): suppose $M = \mathbb{R}^k \times_\Lambda F$. Since $H^1(F;\mathbb{R}) = 0$ it follows from the Leray–Serre spectral sequence that the projection $M \to T^k$ induces an isomorphism $H^1(M;\mathbb{R}) \cong H^1(T^k;\mathbb{R})$. So every harmonic 1-form comes from $T^k$ and hence is parallel.

(2) $\Rightarrow$ (4), cf. Theorem 8.5 in [5] and [3]: suppose $(M,g)$ is a closed, connected and oriented Riemannian manifold, such that every harmonic 1-form is parallel. Choose an orthonormal base $\{\beta_1,\dots,\beta_k\}$ of harmonic 1-forms. Since they are parallel they are orthonormal at every point in $M$. Choose a base point $x_0 \in M$, let $U \subseteq \pi_1(M,x_0)$ be the kernel of the Huréwicz-homomorphism

$$\pi_1(M,x_0) \to H_1(M;\mathbb{Z}) \to \frac{H_1(M;\mathbb{Z})}{\mathrm{Tor}(H_1(M;\mathbb{Z}))} \cong \mathbb{Z}^k,$$

and let $\pi : \tilde{M} \to M$ be the covering of $M$, which has $U$ as characteristic subgroup. This is a normal covering, the group of deck transformations is $\mathbb{Z}^k$ and $\pi^*\beta_i = \mathrm{d}f_i$ for smooth functions $f_i : \tilde{M} \to \mathbb{R}$. Let $z_0$ be a base point in $\tilde{M}$ sitting above $x_0$ and assume $f_i(z_0) = 0$. Consider the mapping

$$f : \tilde{M} \to \mathbb{R}^k, \qquad f(z) = (f_1(z),\dots,f_k(z)).$$

Obviously this is a proper, subjective submersion and $F := f^{-1}(0)$ is a compact submanifold. Let $X_i := \sharp_g\pi^*\beta_i$. Then the $X_i$ are orthonormal at every point and they are all parallel, especially they commute. Consider

$$\kappa : F \times \mathbb{R}^k \to \tilde{M}, \qquad \kappa(z,t) := (\mathrm{Fl}^{X_1}_{t_1} \circ \cdots \circ \mathrm{Fl}^{X_k}_{t_k})(z).$$

Of course we have $f \circ \kappa = \mathrm{pr}_2$, and it follows easily, that $\kappa$ is a diffeomorphism. So $F$ is closed, connected, oriented and $H^1(F;\mathbb{R}) = H^1(\tilde{M};\mathbb{R}) = 0$. Moreover $\kappa^*g$ is the product metric of the induced metric on $F$ and the standard metric on $\mathbb{R}^k$. Every deck transformation of $\tilde{M}$ is of the form

$$F \times \mathbb{R}^k \to F \times \mathbb{R}^k, \qquad (z, t) \mapsto (\varphi_\lambda(z), t + \lambda),$$

where $\lambda \in \Lambda \subseteq \mathbb{R}^k$ and $\varphi_\lambda$ is an orientation preserving isometry of $F$. So $M$ is a twisted product, as claimed.

To see (1) $\Leftrightarrow$ (5) let $\sharp\delta\alpha$ be an exact volume preserving vector field, $\alpha \in \Omega^2(M)$, and let $\beta$ be a harmonic 1-form. Then

$$\int_M g(\mathrm{ad}(\sharp\delta\alpha)^{\mathrm{T}}\sharp\delta\alpha, \sharp\beta)\mu = \int_M g(\sharp\delta\alpha, [\sharp\delta\alpha, \sharp\beta])\mu = -\int_M g(\sharp\delta\alpha, \sharp\delta(\delta\alpha \wedge \beta))\mu$$
$$= -\int_M g(\mathrm{d}\delta\alpha, \delta\alpha \wedge \beta)\mu,$$

where we used

$$\delta(\varphi_1 \wedge \varphi_2) - (\delta\varphi_1) \wedge \varphi_2 + \varphi_1 \wedge \delta\varphi_2 = -\flat[\sharp\varphi_1, \sharp\varphi_2] \quad \text{for} \;\; \varphi_1, \varphi_2 \in \Omega^1(M),$$

to obtain $[\sharp\delta\alpha, \sharp\beta] = -\sharp\delta(\delta\alpha \wedge \beta)$ for the second equality. □

**Proof of Theorem 2.** Recall that the orthogonal complement of $\mathfrak{X}_{\mathrm{ex}}(M, \omega)$ in $\mathfrak{X}(M, \omega)$ is $\{\sharp_\omega\beta : \beta$ harmonic 1-form$\}$. By Remark 1 and Lemma 3, (1) is equivalent to $\nabla(\beta \circ J)^{\mathrm{sym}} = \nabla(\flat_g \sharp_\omega \beta)^{\mathrm{sym}} = 0$ for all harmonic 1-forms $\beta$. On a Kähler manifold one has $\Delta(\varphi \circ J) = (\Delta\varphi) \circ J$ for 1-forms $\varphi$. Particularly the space of harmonic 1-forms is $J$-invariant, and so (1) is equivalent to $(\nabla\beta)^{\mathrm{sym}} = 0$ and since harmonic 1-forms are closed this is equivalent to (2).

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (2) are as in the proof of Theorem 1. For (4) $\Leftrightarrow$ (2) one needs some extra arguments: one observes, that the span of the $X_i$ constructed in the proof of Theorem 1, is $J$-invariant and so is its orthogonal complement. Hence $F$ is a complex submanifold and therefore a Kähler submanifold. Moreover the complex structure is, locally over $T^k$, the product structure and so is the symplectic structure as well.

(1) $\Leftrightarrow$ (5) follows from the following computation for a function $f$ and a closed 1-form $\beta$:

$$\begin{aligned}
&\int_M g(\mathrm{ad}(\sharp_\omega \mathrm{d}f)^{\mathrm{T}}\sharp_\omega \mathrm{d}f, \sharp_\omega\beta)\omega^n \\
&\quad = \int_M g(\sharp_\omega \mathrm{d}f, [\sharp_\omega \mathrm{d}f, \sharp_\omega\beta])\omega^n = -\int_M g(\sharp_\omega \mathrm{d}f, \sharp_\omega \mathrm{d}(L_{\sharp_\omega\beta} f))\omega^n \\
&\quad = -\int_M g(\mathrm{d}f, \mathrm{d}(L_{\sharp_\omega\beta} f))\omega^n \\
&\quad = -\int_M (\Delta f)(L_{\sharp_\omega\beta} f)\omega^n = -n\int_M (\Delta f)\mathrm{d}f \wedge \beta \wedge \omega^{n-1}.
\end{aligned}$$

For the second equality we used $[\sharp_\omega\varphi_1, \sharp_\omega\varphi_2] = -\sharp_\omega(L_{\sharp_\omega\varphi_2}\varphi_1)$ for closed 1-forms $\varphi_1, \varphi_2$, a relation derived from $i_{[X,Y]} = L_X i_Y - i_Y L_X$. □

**Remark 2.** The fact that $M$ is Kähler was only used to show, that the space of harmonic 1-forms is invariant under $J$. In the almost Kähler case the arguments in the proof of Theorem 2 show, that following are equivalent:

1. The group of Hamiltonian diffeomorphisms is a totally geodesic subgroup in the group of all symplectic diffeomorphisms.
2. $\sharp_\omega \beta = \sharp_g(\beta \circ J) = -J\sharp_g \beta$ is Killing for every harmonic 1-form $\beta$.
3. For all functions $f$ and all harmonic 1-forms $\beta$ one has

$$\int_M (\Delta f)\mathrm{d}f \wedge \beta \wedge \omega^{n-1} = 0.$$

**Remark 3.** The computation in the proof of Theorem 2 shows, that for a function $f$, $\mathrm{ad}(\sharp_\omega\, \mathrm{d}f)^{\mathrm{T}}\sharp_\omega\, \mathrm{d}f = 0$ if and only if

$$\int_M (\Delta f)\mathrm{d}f \wedge \beta \wedge \omega^{n-1} = 0 \quad \text{for all closed 1-forms } \beta,$$

even on almost Kähler manifolds. If $f$ is a 'generalized eigenvector' of the Laplacian, i.e. $\Delta f = h \circ f$ for some smooth function $h \in C^\infty(\mathbb{R}, \mathbb{R})$, then

$$\int_M (\Delta f)\mathrm{d}f \wedge \beta \wedge \omega^{n-1} = \int_M (h \circ f)\mathrm{d}f \wedge \beta \wedge \omega^{n-1} = \int_M \mathrm{d}((H \circ f)\beta \wedge \omega^{n-1}) = 0,$$

with $H$ an integral of $h$, consequently the condition is satisfied. So the geodesic is given by an exponential. These are examples of a more general method how to solve the geodesic equation: in the general setting any finite dimensional submanifold $S \subset \mathfrak{g}$ such that $\mathrm{ad}(X)^{\mathrm{T}}X \in T_X S$ for $X \in S$ admits the calculation of flow-lines in the manifold $S$. In the above case the submanifold $S$ is given by a single point.

## 4. Non-existence results

In this paragraph we show there exists no Lie algebra of vector fields $\mathfrak{g}$ containing the Lie algebra of divergence free vector fields or the Lie algebra of symplectic vector fields as totally geodesic Lie subalgebras. For the proof we use the following fact, deduced in Remark 1: if $\mathfrak{h}$ is a totally geodesic splitting Lie subalgebra of a Lie algebra of vector fields $\mathfrak{g}$ and $\mathfrak{h}$ is big enough to detect the vanishing of a symmetric 2-tensor field, then $\mathfrak{h}^\perp$ consists of Killing vector fields (in particular $\mathfrak{h}$ has finite codimension in $\mathfrak{g}$).

**Corollary 1.** *Let $M$ be a closed, connected and oriented Riemannian manifold. Then there does not exist a closed Lie subalgebra $\mathfrak{g}$ satisfying $\mathfrak{X}(M, \mu) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, such that $\mathrm{ad}(\cdot)^{\mathrm{T}} : \mathfrak{g} \to L(\mathfrak{g})$ is bounded and such that $\mathfrak{X}(M, \mu)$ is a totally geodesic Lie subalgebra of $\mathfrak{g}$.*

This follows from Remark 1, since Killing vector fields are divergence free.

**Corollary 2.** *Let $(M, g, J, \omega)$ be a closed, connected almost Kähler manifold. Then the symplectic diffeomorphisms are not totally geodesic in the group of volume preserving diffeomorphisms, provided* $\dim(M) > 2$.

This is also a consequence of Remark 1, since the Lie algebra of symplectic vector fields is not of finite codimension in the Lie algebra of divergence free vector fields.

**Remark 4.** Let $K$ be a compact Lie group acting by isometries on the closed connected orientable manifold $(M, g)$. In [7] it is shown that the group of $K$-equivariant diffeomorphisms is a totally geodesic subgroup of $\mathrm{Diff}(M)$. Its Lie algebra of $K$-invariant vector fields on $M$ is split, a complement is $\{X \in \mathfrak{X}(M) : \int_K k^* X \,\mathrm{d}k = 0\}$, infinite dimensional. This does not contradict the arguments above, since the Lie algebra of $K$-invariant vector fields on $M$ does not detect the vanishing of a symmetric 2-tensor field.

**Remark 5.** The subgroup of diffeomorphisms of $M$ fixing a point $x_0$ is a totally geodesic subgroup of $\mathrm{Diff}(M)$ (see [7]). The subgroup of volume preserving diffeomorphisms fixing a point $x_0 \in M$ has finite codimension (equal to $\dim M$) in $\mathrm{Diff}(M, \mu)$ and its Lie algebra is big enough to detect the vanishing of a symmetric 2-tensor field, but it is not totally geodesic.

A proof, due to Alexander Shnirelman, goes as follows in the case $M$ is a flat torus. If $u_0(x)$ is any divergence free vector field of $M$, and $x_0$ an arbitrary point, we can form a vector field $v_0(x) = u_0(x) - u_0(x_0)$; then $v_0(x_0) = 0$, and if we assume that the group of volume preserving diffeomorphisms of the flat torus fixing a point is totally geodesic in the group of volume preserving diffeomorphisms, the solution $v(x, t)$ with initial condition $v_0(x)$ should satisfy $v(x_0, t) = 0$ for all $t$. A short computation shows that $v(x, t) = u(x + u_0(x_0)t, t) - u_0(x_0)$; thus, the fluid particle which at $t = 0$ was at the point $x_0$ moves with a constant speed $u_0(x_0)$ and the pressure function is constant. But there exist solutions of the Euler equation with non-constant pressure.

Let $\mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M, \omega)$ denote the compactly supported Hamiltonian vector fields.

**Lemma 5.** *Let $(M, \omega)$ be a connected symplectic manifold and let $\alpha \in \Omega^2(M)$. If*

$$L_X \alpha = 0, \quad \text{for all } X \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M, \omega),$$

*then there exists $\lambda \in \mathbb{R}$ such that $\alpha = \lambda\omega$.*

**Proof.** Choose a Darboux chart centered at $z \in M$, such that

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}y^1 + \cdots + \mathrm{d}x^n \wedge \mathrm{d}y^n,$$

and write

$$\alpha = \sum_{i<j} a_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j + \sum_{i<j} b_{ij}\, \mathrm{d}y^i \wedge \mathrm{d}y^j + \sum_{i,j} c_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}y^j.$$

Let $h$ be a compactly supported function on $M$, such that $h = x^i$ resp. $h = y^i$ locally around $z$. Then the condition $L_{\sharp_\omega \mathrm{d}h}\alpha = 0$ shows that $a_{ij}$, $b_{ij}$ and $c_{ij}$ are all constant locally around $z$. Using $h = (x^i)^2$ one sees, that $b_{ij} = 0$ and $c_{ij} = 0$ for $i \neq j$. Using $h = (y^i)^2$ yields $a_{ij} = 0$. Finally, using $h = x^i x^j$ shows $c_{ii} = c_{jj}$. So $\alpha = \lambda\omega$ locally around $z$, for some constant $\lambda \in \mathbb{R}$. Since $M$ is connected this is true globally. □

We denote by

$$\mathfrak{X}(M, [\omega]) := \{X \in \mathfrak{X}(M) : \exists\lambda \in \mathbb{R} : L_X\omega = \lambda\omega\}.$$

Notice that for closed $M$ we have $\mathfrak{X}(M, [\omega]) = \mathfrak{X}(M, \omega)$. Moreover if $L_X\omega = f\omega$ for some function $f$ and if $\dim(M) > 2$ then $f$ is constant, due to the non-degeneracy of $\omega$.

**Lemma 6.** *Let $(M, \omega)$ be a symplectic manifold and let $Z \in \mathfrak{X}(M)$. If*

$$[Z, X] \in \mathfrak{X}(M, [\omega]), \quad \text{for all } X \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M, \omega),$$

*then $Z \in \mathfrak{X}(M, [\omega])$.*

**Proof.** Set $\alpha := L_Z\omega \in \Omega^2(M)$. Then for every $X \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M, \omega)$ we have

$$L_X\alpha = L_X L_Z\omega = L_{[X,Z]}\omega = \lambda\omega = 0.$$

Here $\lambda$ has to vanish, since $[X, Z]$ has compact support. So by the previous lemma there exists $\tilde{\lambda} \in \mathbb{R}$, such that $L_Z\omega = \alpha = \tilde{\lambda}\omega$, i.e. $Z \in \mathfrak{X}(M, [\omega])$. □

**Proposition 1.** *Let $(M, \omega)$ be a symplectic manifold. Then there does not exist a Lie subalgebra $\mathfrak{g}$ satisfying $\mathfrak{X}(M, [\omega]) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, such that $\mathfrak{X}(M, [\omega])$ has finite codimension in $\mathfrak{g}$.*

**Proof.** Suppose $\mathfrak{g}$ is bigger than $\mathfrak{X}(M, [\omega])$. Then there exists $Z \in \mathfrak{g}$ and an open subset $U \subseteq M$, such that $Z|_V \notin \mathfrak{X}(V, [\omega])$, for every all open $V \subseteq U$. For any $k \in \mathbb{N}$ we choose disjoint subsets $V_1, \dots, V_k \subseteq U$. Since $Z|_{V_i} \notin \mathfrak{X}(V_i, [\omega])$ Lemma 6 yields $X_i \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(V_i, \omega)$, such that $Y_i := [Z, X_i] \notin \mathfrak{X}(V_i, [\omega])$. But $Y_i \in \mathfrak{g}$ and obviously $\{Y_1, \dots, Y_k\}$ are linearly independent in $\mathfrak{g}/\mathfrak{X}(M, [\omega])$. Hence the codimension of $\mathfrak{X}(M, [\omega])$ in $\mathfrak{g}$ is at least $k$. Since $k$ was arbitrary we are done. □

**Corollary 3.** *Let $(M, g, J, \omega)$ be a closed, connected almost Kähler manifold. Then there does not exist a closed Lie subalgebra $\mathfrak{g}$ satisfying $\mathfrak{X}(M, \omega) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, such that $\mathrm{ad}(\cdot)^{\mathrm{T}} : \mathfrak{g} \to L(\mathfrak{g})$ is bounded and such that $\mathfrak{X}(M, \omega)$ is totally geodesic in $\mathfrak{g}$.*

**Proof.** Suppose conversely such a $\mathfrak{g}$ exists. By Remark 1 and Lemma 3, $\mathfrak{X}(M, \omega)$ has an orthogonal complement in $\mathfrak{g}$, consisting of Killing vector fields. So this complement has to be finite dimensional, but this contradicts Proposition 1. □

For a manifold with volume form $(M, \mu)$ we let

$$\mathfrak{X}(M, [\mu]) := \{X \in \mathfrak{X}(M) : \exists\lambda \in : L_X\mu = \lambda\mu\}.$$

Notice, that for closed $M$ one has $\mathfrak{X}(M, \mu) = \mathfrak{X}(M, [\mu])$. Similarly, although it does not yield anything new for our totally geodesic subgroups, one shows.

**Proposition 2.** *Let* $(M, \mu)$ *be a manifold with volume form and* $\dim(M) > 1$. *Then there does not exist a Lie subalgebra* $\mathfrak{g}$ *satisfying* $\mathfrak{X}(M, [\mu]) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, *such that* $\mathfrak{X}(M, [\mu])$ *has finite codimension in* $\mathfrak{g}$.

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## References

[1] V.I. Arnold, Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier 16 (1966) 319–361.
[2] V.I. Arnold, B.A. Khesin, Topological Methods in Hydrodynamics, Springer, Berlin, 1998.
[3] J. Cheeger, D. Gromoll, The splitting theorem for manifolds of nonnegative ricci curvature, J. Diff. Geom. 6 (1971) 119–128.
[4] A. Kriegl, P.W. Michor, The convenient setting of global analysis, Mathematical Surveys and Monographs 53, AMS, Providence, RI, 1997.
[5] H.B. Lawson, M.-L. Michelson, Spin Geometry, Princeton University Press, Princeton, NJ, 1989.
[6] P.W. Michor, T. Ratiu, On the geometry of the Virasoro–Bott group, J. Lie Theory 8 (1998) 293–309.
[7] C. Vizman, Geodesics and curvature of diffeomorphism groups, in: Proceedings of the Fourth International Workshop on Differential Geometry, Brasov, Romania, 1999, pp. 298–305.